A Proof that Kramer's Multiple Comparison Procedure for Differences Between Treatment Means is Level-α for 3,4, or 5 Treatments

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Summary

Let X_{ij}, $i = 1,\ldots,p$; $j = 1,\ldots,N_i$ be independent normal variables with $E(X_{ij}) = \mu_i$, $\mathrm{Var}\, X_{ij} = \sigma^2$. Let $X_{i.} = N_i^{-1} \sum_{j=1}^{N_i} X_{ij}$ and $S^2 = (\Sigma(N_i-1))^{-1}\, \Sigma(X_{ij}-X_{i.})^2$. Then if $p \leqq 5$ we show

$$\Pr(|(\mu_i-\mu_j)-(X_{i.}-X_{j.})| \geqq S(N_i^{-1}+N_j^{-1})q_{p,\nu}^{(\alpha)}/2^{1/2} \text{ for some } i \neq j)$$

$$\leqq \alpha$$

where $q_{p,\nu}^{(\alpha)}$ denotes the upper α^{th} quantile of the Studentized range distribution. This validates the use of Kramer's multiple comparison procedure (proposed in Kramer, C.Y. (1956). Extension of multiple range tests to group means with unequal numbers of replications. Biometrics 12, 307-310) when $p \leqq 5$. (The result for $p = 3$ was previously proven in Kurtz, T.E. (1956). An extension of a multiple comparison procedure. Doctoral thesis, Princeton University.)

1. Introduction

Kramer (1956, 1957) proposed the following multiple comparison procedure for a one-way analysis of variance with unequal sample sizes.

Let X_{ij}, $i = 1,\ldots,p$; $j = 1,\ldots,N_i$ $(N_i \geq 1)$ be independent normal variables with $E(X_{ij}) = \mu_i$ $\operatorname{Var} X_{ij} = \sigma^2$. As usual let $X_{i.} = N_i^{-1} \sum_{i=1}^{N_i} X_{ij}$ and $S^2 = \nu^{-1}\Sigma(X_{ij}-X_{i.})^2$ where $\nu = N-p = (\Sigma N_i)-p \geq 1$. Kramer's procedure produces the set of simultaneous confidence statements

$$(1.1)\qquad |(\mu_i-\mu_j)-(x_{i.}-x_{j.})| \leq s(N_i^{-1}+N_j^{-1})^{1/2} q_{p,\nu}^{(\alpha)}/2^{1/2} ,$$

where $q_{p,\nu}^{(\alpha)}$ is the upper α^{th} quantile of the Studentized range distribution (see e.g. Miller (1966), p.37-47). The conjecture, on which Kramer's proposal is based is that the overall error rate of (1) as α. A precise statement of this conjecture is as follows:

<u>Conjecture 1</u>: Let p be given. Then for any configuration N_i, $i = 1,\ldots,p$

$$(1.2)\qquad \Pr(|(X_{i.}-X_{j.})-(\mu_i-\mu_j)| > S(N_i^{-1}+N_j^{-1})^{1/2} q_{p,\nu}^{(\alpha)}/2^{1/2} \text{ for some } i \neq j)$$
$$\leq \alpha.$$

(By the definition of $q_{p,\nu}^{(\alpha)}$ equality is achieved in (1.2) when $N_i \equiv N/p$.)

The validity of Conjecture 1 would show, in other words, that Kramer's procedure is conservative. This conjecture is trivial for $p = 2$ and was verified for $p = 3$ in Kurtz (1956). However its validity for $p \geq 4$ has not previously been validated. As pointed out in Miller (1966), (See, e.g., p.87.) this procedure can only be used with skepticism unless Conjecture 1 is validated. Many related references are listed in Miller (1977) and some related issues have recently been discussed in Gabriel (1978) and Genizi and Hochberg (1978).

Recently Dunnett (1979) has performed a careful Monte Carlo study which indicates - perhaps somewhat surprisingly - that Conjecture 1 may be valid for all values of p. Dunnett's study provides strong evidence, and may suffice to justify the use of Kramer's procedure in various applications - particularly when p is not large. However a Monte Carlo study is not well suited to demonstrating the mathematical validity of Conjecture 1 for several reasons; not least because the Monte Carlo calculations must be performed at all configurations of N_i, $i = 1,\ldots,p$, and this is increasingly impossible to do as p gets large.

Motivated by Dunnett's study we have attempted to find a proof of Conjecture 1 for values of $p \geq 4$. We have had only limited success. This paper contains a proof of

<u>Theorem 1</u>: Conjecture 1 is valid for $p = 4$ and for $p = 5$ (as well as $p = 3$).

The fact that Theorem 1 is limited to $p \leq 5$ has not made us skeptical of the validity of Conjecture 1 for larger values of p. Rather it has convinced us only that the problem is mathematically very complex to solve - at least via the methods which we have found to be applicable and have used here. These methods are entirely elementary and it could be that the problem will prove more tractable via some more sophisticated approach. Still, there is some hope that the methods used here could be extended to yield a general proof of Conjecture 1 (if that conjecture is really valid for all p). It would be necessary to enunciate a suitable induction hypothesis; and this we have been unable to do here. Some further thoughts on this issue are contained in a postscript in the Appendix.

The validity of Conjecture 1 is equivalent to the validity of the following proposition concerning a family Z_i, $i = 1,\dots,p$, of standard (mean zero, variance 1) normal random variables.

For $\rho = (\rho_1,\dots,\rho_p)$ with $\rho_i > 0$ define

$$P_k(\rho) = \Pr\{|\rho_i Z_i - \rho_j Z_j| > k(\rho_i^2+\rho_j^2)^{1/2} \text{ for some } i \neq j\}.$$

<u>Proposition 1</u>: Let p be given. Then for any $k > 0$

$$\begin{aligned}(1.3)\qquad & \sup\{P_k(\rho):\rho = (\rho_1,\dots,\rho_p) \text{ with } \rho_i > 0\} \\ &= \Pr\{|Z_i - Z_j| > 2^{1/2}k \text{ for some } i \neq j\} \\ &= P_k(\underline{\underline{1}}).\end{aligned}$$

(See the appendix for a proof that Proposition 1 implies Conjecture 1. The reverse implication is of no importance here, and its proof is left to the reader.)

2. Reduction to a local maxima problem.

Our proof of the validity of Proposition 1 involves showing for the given dimension, p, that the probability on the left of (1.3) is a continuously differentiable function of $\rho = (\rho_1, \ldots, \rho_p)$ which has an inflection point if and only if $\rho_1 = \ldots = \rho_p$. It is then easy to show that this inflection point is a local and global maximum, and hence Proposition 1 (and Conjecture 1) is valid for this value of p.

Let e_i denote the ith unit vector in $\mathbb{R}^p$. Let $\beta_{ij}(\rho) = (\rho_i e_i - \rho_j e_j)/(\rho_i^2 + \rho_j^2)^{1/2}$. Let $Z = (Z_1, \ldots, Z_p) \in \mathbb{R}^p$ and use the symbols $\alpha \cdot \beta$ for the ordinary dot product on $\mathbb{R}^p$ and $\|\alpha\| = (\alpha\cdot\alpha)^{1/2}$. For any collection $\beta = \{\beta_{ij}: \beta_{ij} \in \mathbb{R}^p, 1 \leqq i < j \leqq p\}$ define

$$\Pi_k(\beta) = \Pr\{|\beta_{ij}\cdot Z| > k\|\beta_{ij}\| \text{ for some } i < j\}. \tag{2.1}$$

Note that $P_k(\rho) = \Pi_k(\beta(\rho))$ where $P_k(\rho)$ is defined in Proposition 1. When the value of k is fixed in advance, as it will be in the following, we write $P(\rho)$ in place of $P_k(\rho)$, etc. It is easy to check that $\Pi(\beta)$ and $\beta(\rho)$ are continuously differentiable to all orders.

Then,

$$\frac{\partial}{\partial \rho_p} \Pi(\beta(\rho)) = \sum_{i=1}^{p-1} \frac{\partial \Pi(\beta)}{\partial \beta_{ip}} \cdot ({}^{e_p}/_{\rho_p} + {}^{e_i}/_{\rho_i})\left(-\frac{\rho_i^2 \rho_p}{(\rho_i^2+\rho_p^2)^{3/2}}\right) \tag{2.2}$$

$$= \sum_{i=1}^{p-1} \Gamma_i(\rho) \text{ (definition)}$$

where

$$\left(\frac{\partial \Pi(\beta)}{\partial \beta_{ip}}\right)_m = \lim_{\Delta \to 0} \Delta^{-1} \{\Pi(\beta + \Delta\, e^*_{i,p}(m)) - \Pi(\beta)\}$$

with $(e^*_{\hat{i},p}(m))_{ij} = e_m$ if $i = \hat{i}$, $j = p$, and $= 0$ otherwise.

We thus intend to prove

Proposition 2: Let p be given. Let $\rho_p \geqq \rho_i$ for all $i < p$. Then

$$\Gamma_i(\rho) \leqq 0 \tag{2.3}$$

for all $i < p$. Equality occurs (if and) only if $\rho_i = \rho_p$.

According to the introductory remarks of this section (including (2.2) the validity of Proposition 2 will imply the validity of Proposition 1 (and of Conjecture 1). Note, it is essential in Proposition 2 that $\rho_p \geqq \rho_i$ for $i < p$; otherwise (2.3) could be false. In more concrete terms Proposition 2 implies that $\Pi(\beta(\rho))$ can always be increased by slightly decreasing the largest value of ρ_i unless all ρ_i are equal. (This implication is valid even if several (but not all) coordinates of ρ all assume the value $\max_i \rho_i$.)

The next step is to give a more useful expression for $\Gamma_i(\rho)$. To this end, assume $\rho_p \geqq \rho_i$ for $i < p$. Without loss of generality fix the particular value of the index i which appears on the left of (2.3) at $i = p-1$. Define

$$V_i = -\rho_i Z_i \quad i = 1,\ldots,p-2$$

$$V_{p-1} = -(\rho_{p-1} e_p + \rho_p e_{p-1}) \cdot Z/(\rho_p^2 + \rho_{p-1}^2)^{1/2}$$

$$V_p = (\rho_{p-1} e_{p-1} - \rho_p e_p) \cdot Z/(\rho_p^2 + \rho_{p-1}^2)^{1/2}$$

$$= \beta_{p-1,p}(\rho) \cdot Z$$

$$(\beta^*(\rho,\Delta))_{p-1,p} = \beta_{p-1,p}(\rho + \Delta e_p)$$

$$(\beta^*(\rho,\Delta))_{ij} = \beta_{ij}(\rho) \quad (i,j) \neq (p-1,p);\ i < j .$$

Note that V_i are independent normal variables; and that V_p and V_{p-1} have variance one. Now,

$$(2.4) \qquad (\beta^*(\rho,\Delta))_{p-1,p}\cdot Z = \beta_{p-1,p}(\rho + \Delta e_p)\cdot Z = V_p + \Delta \rho_{p-1} V_{p-1}/(\rho_p^2 + \rho_{p-1}^2) + o_p(\Delta).$$

(A more detailed presentation of (2.4)-as well as many of the following expressions-appears in the appendix.). Then

$$(2.5) \qquad \Gamma_{p-1}(\rho) = \lim_{\Delta\to 0} \Delta^{-1} [\Pr(\max\{|\beta^*_{ij}(\rho,\Delta)\cdot Z|\} \geqq k) - \Pr(\max\{|\beta_{ij}(\rho)\cdot Z|\} \geqq k)]$$

$$= 2\varphi(k)\, E((\rho_{p-1} V_{p-1}/(\rho_p^2 + \rho_{p-1}^2))^+ (\Pr(\hat{S}(V_{p-1})) - \Pr(\hat{S}(-V_{p-1})))|V_p = k).$$

where φ denotes the standard normal density function, $a^+ = \max(a,0)$, and

$$\hat{S}(v) = \{(v_1,\dots,v_{p-2}): \max\{|\beta_{ij}\cdot Z| \; (i,j) \neq (p-1,p)\} < k \mid V_p = k,\; V_{p-1} = v).$$

After some algebraic calculations it is possible to write $\hat{S}(v) = S(\rho_p\rho_{p-1} v/(\rho_p^2 + \rho_{p-1}^2)^{1/2})$ where

$$(2.6) \qquad S(a) = \{(v_1,\dots,v_{p-2}): |v_i - v_j| < k(\rho_i^2 + \rho_j^2)^{1/2}, \quad a + c_i - d_i < v_i < a + c_i + d_i\} \text{ with}$$

$$c_i = (k/2)((\rho_{p-1}^2 + \rho_i^2)^{1/2} - (\rho_p^2 + \rho_i^2)^{1/2} + (\rho_p^2 - \rho_{p-1}^2)/(\rho_p^2 + p_{p-1}^2)^{1/2})$$

and

$$d_i = (k/2)((\rho_p^2 + \rho_i^2)^{1/2} + (\rho_{p-1}^2 + \rho_i^2)^{1/2} - (\rho_p^2 + \rho_{p-1}^2)^{1/2}).$$

Note that since $\rho_p \geqq \rho_{p-1}$

$$c_i \geqq (k/2)((\rho_{p-1}^2 + \rho_i^2)^{1/2} - (\rho_p^2 + \rho_i^2)^{1/2} + (\rho_p - \rho_{p-1})) \geqq 0 \tag{2.7}$$

(because $(y^2 + \rho_i^2)^{1/2} - y$ is a decreasing function of $y \geqq 0$). Note also that $c_i = 0$ if and only if $\rho_p = \rho_{p-1}$. Also note that c_i and d_i are increasing functions of ρ_i for fixed ρ_p, ρ_{p-1}. See Figure 2.1 for a picture of $S(\pm a)$ when $p = 4$. [Insert Figure 2.1 here.]

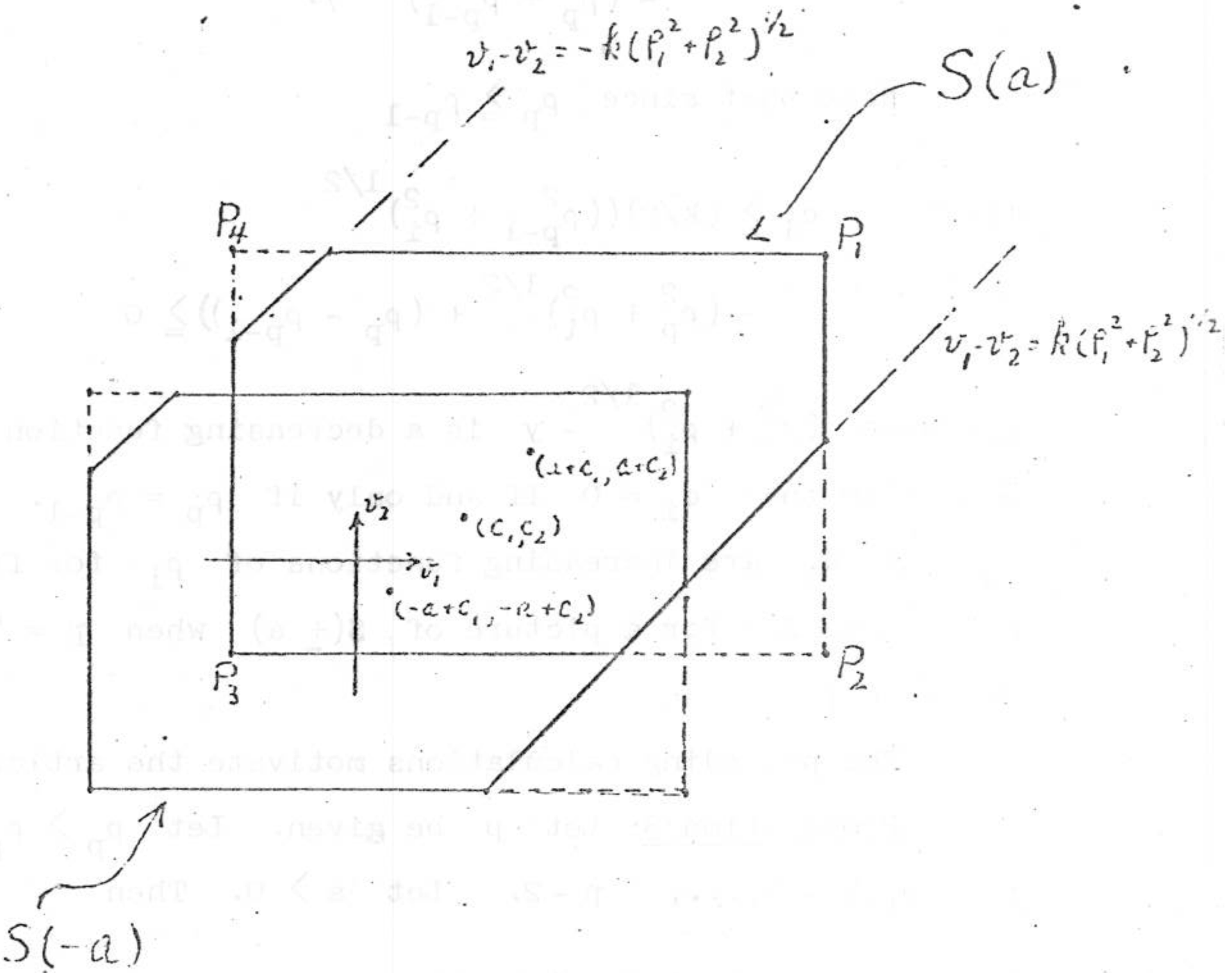
The preceding calculations motivate the articulation of

<u>Proposition 3</u>: Let p be given. Let $\rho_p > \rho_{p-1}$ and $\rho_p \geqq \rho_i, i = 1, \ldots, p-2$. Let $a > 0$. Then

$$\Pr(S(a)) < \Pr(S(-a)). \tag{2.8}$$

It follows by continuity that $\Pr(S(a)) \leqq \Pr(S(-a))$ when $\rho_p = \rho_{p-1}$. (In fact, later computations show that equality holds under this condition.)

According to the calculations preceding this proposition <u>the truth of Proposition 3 for the given p implies the truth of Proposition 2, and consequently of Conjecture 1, for this value of p.</u>



$v_1 - v_2 = -k(\rho_1^2 + \rho_2^2)^{1/2}$
S(a)
P_4
P_1
$v_1 - v_2 = k(\rho_1^2 + \rho_2^2)^{1/2}$
$(a+c_1, a+c_2)$
v_2
(c_1, c_2)
v_1
$(-a+c_1, -a+c_2)$
P_3
P_2
S(-a)

Fig. 2.1

3. Proof of Proposition 3 (and Conjecture 1) for $p = 3$.

When $p = 3$ then $S(a) = \{v_1 : a + c_i - d_1 < v_1 < a + c_1 + d_1\}$. Now, let W_1 be a normal variable with mean 0 and variance $\rho_1^2 > 0$. Then $\pi(b) = \Pr\{b - d_i < W_1 < b + d_i\}$ is a symmetric strictly unimodel function. Hence

$$(3.1) \qquad \Pr(S(-a)) = \pi(-a + c_1)$$
$$> \pi(a + c_1) = \Pr(S(a)).$$

This proves Proposition 3 and hence Conjecture 1 for $p = 3$. [Conjecture 1 for $p = 3$ was first proved by Kurtz (1956). His proof was somewhat different, and did not involve intermediate steps like Propositions 1-3.]

The inequality (3.1) can be thought of as an instance of the following standard lemma, which we shall later use again.

Lemma 3.1: Let W be a real valued normal random variable with mean θ, variance σ^2. Let $\alpha < \beta$, with $(\alpha + \beta)/2 \geqq 0$ ($\leqq 0$, respectively) If $\theta \geqq 0$ ($\theta \leqq 0$) then

$$(3.2) \qquad \Pr_\theta((\alpha, \beta)) \geqq \Pr_{-\theta}((\alpha, \beta)) = \Pr_\theta((-\beta, -\alpha))$$

with equality if and only if $(\alpha + \beta)/2 = 0$ or $\theta = 0$.

(In (3.1) $\alpha = c_1 - d_1$, $\beta = c_1 + d_1$, $\theta = a$, and $\Pr(S(-a)) = \Pr_\theta((\alpha, \beta)) > \Pr_{-\theta}((\alpha, \beta)) = \Pr(S(a))$.)

4. Two Lemmas:

The results of this section will be used in the next two sections in the proof of Conjecture 1 for $p = 4,5$. The first of these lemmas is a generalization of Lemma 3.1 to the case of higher dimension, as well as to a more general type of region.

The second lemma is somewhat more specialized (and easier to prove) and will be used several times in Section 6.

Lemma 4.1: Let $W_1,\ldots,W_q$ be independent normal random variables with means $\theta_i \leqq 0$. Let α_i, β_i, $i = 1,\ldots,q$ satisfy $\alpha_i + \beta_i \leqq 0$, $\alpha_i < \beta_i$. Let $k_{ij} > 0$ for $1 \leqq i < j \leqq q$, and define

$$(4.1) \quad R^- = \{w\colon \alpha_i - a < w_i < \beta_i - a,\ldots,q;\ |w_i - w_j| < k_{ij}, 1 \leqq i < j \leqq q\}$$

$$(4.2) \quad R^+ = \{w\colon -\beta_i + a < w_i < -\alpha_i + a,\ i = 1,\ldots,q;\ |w_i - w_j| < k_{ij},\ 1 \leqq i < j \leqq q\}.$$

Then

$$Pr_\theta(R^-) \geqq Pr_\theta(R^+)$$

with equality if and only if $\theta = 0$ or $R^+ = R^-$.

Proof: For the case $q = 1$ this lemma is equivalent to Lemma 3.1. Now, suppose the Lemma is true in dimension $q = 1,\ldots,Q-1$. Let $q = Q$.

Suppose $\beta_j \leqq 0$ for some coordinate $j = 1,\ldots,Q$. For simplicity suppose $j = Q$ so that $\beta_Q \leqq 0$. Then $R^{\pm} = \underset{\omega\in(-\beta_Q,\alpha_Q)}{U}(R^{\pm}(\pm\omega) \times (\pm\omega))$ where

$$\begin{aligned} R^-(-\omega) &= \{(w_1,\ldots,w_{Q-1})\colon (w_1,\ldots,w_{Q-1},-\omega) \in R^-\} \\ &= \{(w_1,\ldots,w_{Q-1})\colon \max(\alpha_i, -\omega-k_{i1}) \\ &< w_i < \min(\beta_i, -\omega+k_{1i})\}, \end{aligned}$$

and $R^+(\omega) = - R^-(-\omega)$. Note that

$$Pr_\theta(R^-(-\omega)) \geqq Pr_\theta(R^+(+\omega))$$

by the induction hypothesis. Hence

$$(5.3) \qquad \begin{aligned} Pr_\theta(R^-) &= E_\theta(Pr_\theta(R^-(-\omega)|W_Q = -\omega)) \\ &\geqq E_\theta(Pr_\theta(R^+(+\omega)|W_Q = -\omega)) \\ &\geqq E_\theta(Pr_\theta(R^+(\omega)|W_Q = \omega)) = Pr_\theta(R^+) \end{aligned}$$

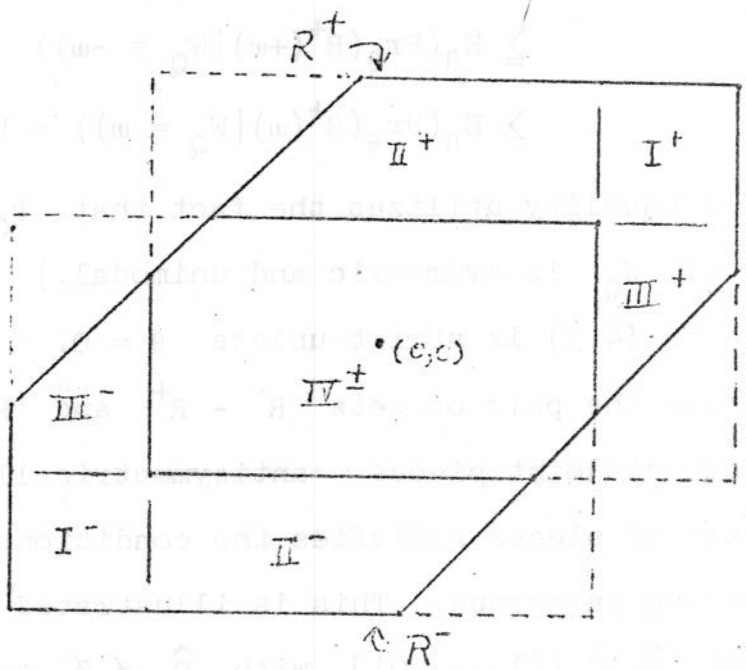
(The second inequality utilizes the fact that $\theta_Q \leqq 0$, $\omega > 0$, and the density of W_Q is symmetric and unimodal.) At least one of the inequalities in (4.3) is strict unless $\theta = 0$.

In general the pair of sets $R^- - R^+$ and $R^+ - R^-$ can be broken up into disjoint pieces - antisymmetrically paired - such that each pair of pieces satisfies the conditions of the lemma and of the preceding paragraph. This is illustrated in Figure 4.1. Explicitly, let $\hat{Q}_1 \subset \{(1,\ldots,Q)\}$ with $\hat{Q}_1 \neq \emptyset$ and let $\hat{Q}_2 = \{(1,\ldots,Q)\} - \hat{Q}_1$. Let

$$\begin{aligned} R^-(\hat{Q}_1) = \{w\colon w \in R^-, \alpha_i-a \\ < w_i < a-\beta_i \text{ for } i \in \hat{Q}_1, \\ \alpha-\beta_i < w_i < \beta_i-a \text{ for } i \in \hat{Q}_2\}. \end{aligned}$$

and define $R^+(\hat{Q}_1) = - R^-(\hat{Q}_1)$ then $R^\pm - R^\mp = \bigcup_{\hat{Q}_1 \neq \emptyset} R^\pm(\hat{Q}_1)$ (except for a set of measure zero) and $Pr_\theta(R^-(\hat{Q}_1)) \geqq Pr_\theta(R^+(\hat{Q}_1))$ with strict inequality unless $R^\pm(\hat{Q}_1) = \emptyset$ or $\theta = 0$. The truth of the lemma for $q = Q$ follows directly. This completes the proof of the lemma.

<u>Lemma 4.2</u>: Let W_1, W_2 be independent normal random variables with mean θ_1, θ_2. Let $R(a)$ be a set of the form



$I^{\pm} = R^{\pm}(\{1,2\})$, $II^{\pm} = R^{\pm}(\{1\}$, $III^{\pm} = R^{\pm}(\{2\})$

$IV^{\pm} = R^{\pm}(\emptyset)$; $R^{\pm} = I^{\pm} \cup II^{\pm} \cup III^{\pm} \cup IV^{\pm}$

Fig. 4.1

$$(4.4) \qquad R(a) = \{(w_1,w_2):\ \alpha_1 + a < w_1 < \beta_1 + a$$
$$\alpha_2 < w_2 - \gamma w_1 < \beta_2\}$$

where $\alpha_i < \beta_i$, $i = 1,2$, $\gamma > 0$, $\alpha_1 + \beta_1 \geqq 0$, $\alpha_2 + \beta_2 + \gamma(\alpha_1+\beta_1) \geqq 0$. Then

$$\Pr_\theta(R(-a) \geqq \Pr_\theta(R(+a))$$

whenever $\theta_1 \leqq 0$, $\theta_2 \leqq 0$; and strict inequality holds unless $\alpha_i + \beta_i = 0$, $i = 1,2$, and $\theta = 0$.

Proof: The sets $R(\pm a)$ are illustrated in Figure 4.2.

Note that for any $\omega > 0$

$$\begin{aligned}\Pr_\theta(\alpha_2+\gamma((\alpha_1+\beta_1)/2-\omega) &< W_2 \\ &< \beta_2 + \gamma((\alpha_1+\beta_1)/2-\omega)) \\ &\geqq \Pr_\theta(\alpha_2+\gamma((\alpha_1+\beta_1)/2+\omega) < W_2 \\ &< \beta_2 + \gamma((\alpha_1+\beta_1)/2+\omega)\end{aligned}$$

by Lemma 3.1 since $\alpha_1 + \beta_1 + \gamma(\alpha_2+\beta_2) \geqq 0$. Hence

$$(4.5) \qquad \Pr_\theta(R(-a)|W_1 = (\alpha_1+\beta_1)/2-\omega)$$
$$\geqq \Pr_\theta(R(+a)|W_1 = (\alpha_1+\beta_1)/2+\omega)$$

for any $\omega \geqq 0$; and strict inequality holds unless $\alpha_2 + \beta_2 + \gamma(\alpha_1+\beta_1)=($ and $\theta_2 = 0$.

Note that

$$(4.6) \quad R(\pm a) - R(\mp a) = R(\pm a) \cap \{(w_1,w_2):\ w_1 = (\alpha+\beta)/2 \pm \omega$$
$$\text{with } |(\alpha_1-\beta_1)/2+a| < \omega < (\beta_1-\alpha_1)/2+a)\}$$

Since the density of W_1 is symmetric and unimodal about $\theta_2 \leqq 0$ it follows from (4.5) and (4.6) that for $\theta \leqq 0$

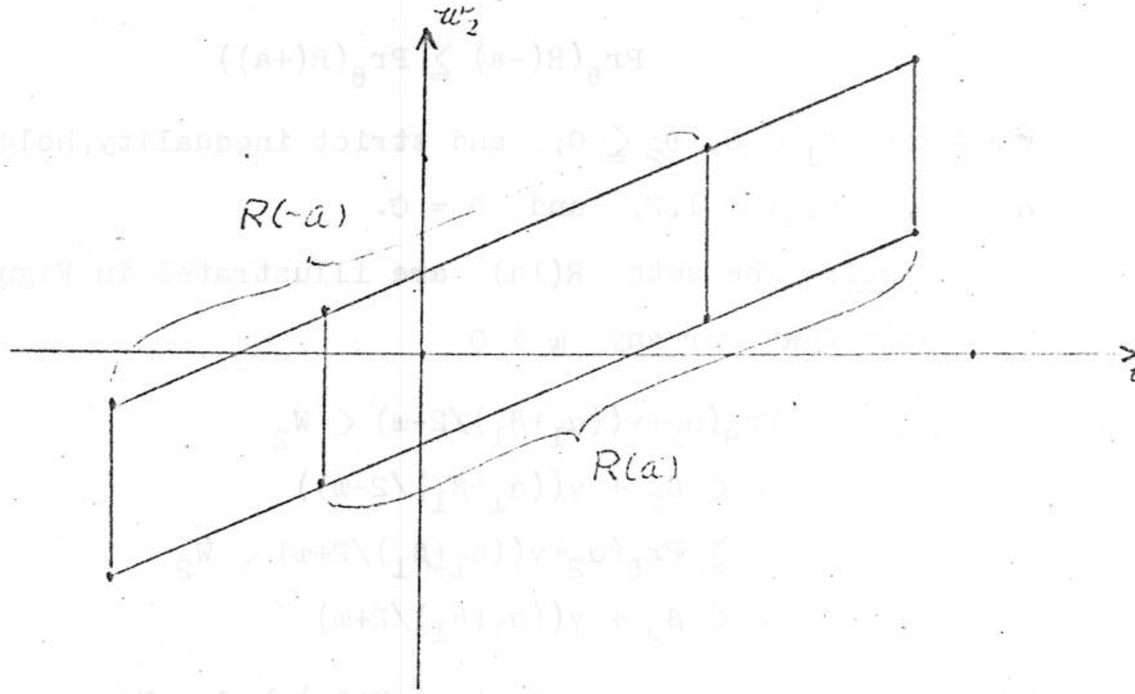


Fig. 4.2

$$\Pr_\theta(R(-a) - R(+a)) \geqq \Pr_\theta(R(+a) - R(-a));$$

with strict inequality unless $\alpha_i + \beta_i = 0$, $i = 1,2$ and $\theta = 0$. This completes the proof of the lemma.

We remark that the preceding lemma could be generalized to hold for sets of the form

$$R(a) = \{(w_1,\ldots,w_q): \alpha_1 + a < w_1 < \beta_1 + a,$$
$$(w_2,\ldots,w_q) \in R^*(w_1)\}$$

so long as the analog of (4.5) holds - that is, so long as

$$(4.7) \qquad \Pr_\theta(R^*(w_1) | W_1 = w_1 = (\alpha_1+\beta_1)/2-\omega)$$
$$\geqq \Pr_\theta(R^*(w_1) | W_1 = w_1 = (\alpha_1+\beta_1)/2+\omega)$$

for all $\omega > 0$.

5. Proof of Proposition 3 (and Conjecture 1) for $p = 4$.

Without loss of generality assume $\rho_1 \geq \rho_2$ throughout this section; as well $p = 4$ and the remaining assumptions of Proposition 3.

The proof is in two parts. The first part concerns the sets

$$S_1(a) = \{(v_1, v_2) \in S(a):$$
$$- k(\rho_1^2+\rho_2^2)^{1/2} + 2(c_1-c_2) \leq v_1 - v_2 \leq k(\rho_1^2+\rho_2^2)^{1/2}\}$$

and the second part concerns the sets

$$S_2(a) = S(a) - S_1(a).$$

Figure 5.1 illustrates these two sets. [Insert Figure 5.1 near here.] It should be clear that the sets $S_1(a)$ and $S_1(-a)$ are anti-symmetric about the point $v = c$. Thus

$$(5.1) \qquad (v_1, v_2) \in S_1(\pm a) \Longleftrightarrow$$
$$2c - (v_1, v_2) \in S_1(\mp a)$$

In fact, S_1 is the maximal set having this property. Thus

$$(5.2) \qquad (v_1, v_2) \in S(\pm a),\ 2c - (v_1, v_2) \in S(\mp a)$$
$$\Longleftrightarrow (v_1, v_2) \in S_1(\pm a).$$

At this stage we have found it notationally and conceptually convenient to introduce independent normal random variables W_1, W_2 with means θ_1, θ_2 and variances ρ_1^2, ρ_2^2. Let

$$T(a) = \{(w_1, w_2):\ w - c = (w_1-c_1, w_2-c_2) \in S(a)\}$$
$$= \{(w_1, w_2):\ |w_1-w_2-(c_1-c_2)|$$
$$< k(\rho_1^2+\rho_2^2)^{1/2};\ |w_i-a| < d_i,\ i = 1,2\}$$

and let

$$T_i(a) = \{(w_1,w_2): w - c \in S_i(a)\},\ i = 1,2.$$

Note that

$$(5.3) \qquad Pr_{\theta=-c}(W \in T(a)) = Pr(V \in S(a))$$

and similarly for $T_i(a)$ and $S_i(a)$, $i = 1,2$.

The first part of the proof of Proposition 3 is contained in the following lemma.

Lemma 5.1: If $\theta_i \leq 0$, $i = 1,2$, then $Pr_\theta(T_1(-a)) \geq Pr_\theta(T_1(a))$, with equality if and only if $\theta = 0$. Consequently $Pr(S_1(-a)) > Pr(S_1(a))$.

Proof: The second sentence of the lemma follows directly from the first, (5.3), and (2.7). To validate the first, note that

$$T_1(\pm a) = \{(w_1,w_2): -d_i \pm a < w_i < d_i \pm a,\ i = 1,2;\ |w_1-w_2| < k(\rho_1^2+\rho_2^2)^{1/2} - (c_1-c_2)\}.$$

Hence $T_1(\pm a)$ are sets of the form $R^\pm$ of Lemma 4.1. Lemma 5.1 now follows from Lemma 4.1.

The reader may note that the preceding lemma does not use many of the detailed formula defining $S_1(\pm a)$ (and, consequently, $T_1(\pm a)$). It uses primarily the antisymmetry, the fact that $c_i > 0$, $i = 1,2$, and the independence and unimodality of the distributions of V_1, V_2 (and W_1, W_2). The next lemma, which deals with the sets $S_2(\pm a)$ requires much more precise information for its validity.

Lemma 5.2: $Pr(S_2(-a)) \geq Pr(S_2(a))$. (It can be shown that equality holds only if $S_2(\pm a) = \emptyset$; which occurs if and only if

$\rho_1 = \rho_2$ (so that $c_1 = c_2$), or $\rho_3 \leq \rho_2$--see the part of the Appendix concerning Figure 2.1.)

Proof: Let $Y_1 = V_1 - V_2$, $Y_2 = V_1/\rho_1^2 + V_2/\rho_2^2$. Note that Y_1, Y_2 are independent normal random variables each having mean 0. Let

$$U_2(\pm a) = \{(y_1, y_2): (v_1(y_1, y_2), v_2(y_1, y_2)) \in S_2(\pm a)$$

Here, of course, $v_1(y_1, y_2) = (\rho_1^2 y_1 + \rho_1^2 \rho_2^2 y_2)/(\rho_1^2 + \rho_2^2)$, and $v_2(y_1, y_2) = (\rho_1^2 \rho_2^2 y_2 - \rho_2^2 y_1)/(\rho_1^2 + \rho_2^2)$.

Now, U_2 can be written as

$$(5.4)\quad U_2(\pm a) = \{(y_1, y_2): \max(-d_1 - d_2 + c_1 - c_2, -k(\rho_1^2 + \rho_2^2)^{1/2}) < y_1 < -k(\rho_1^2 + \rho_2^2) + 2(c_1 - c_2) \text{ and } \eta_1(y_1) \pm a < y_2 < \eta_2(y_1) \pm a\}.$$

See Figure 5.2. It is calculated in the appendix that

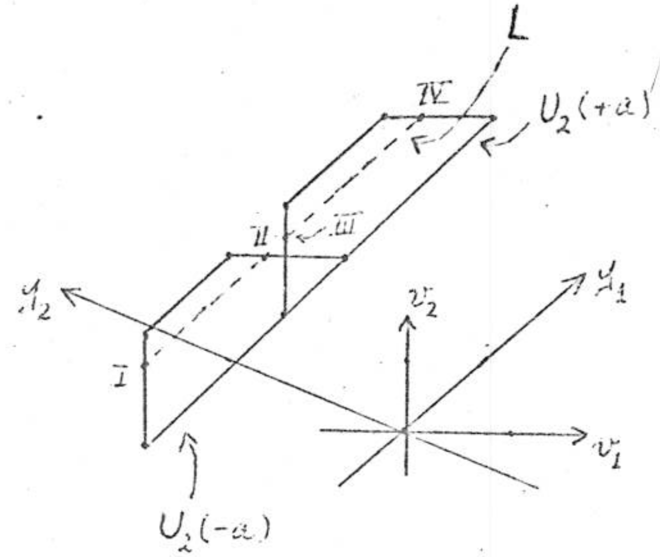
$$(5.5)\qquad (\eta_1(y_1) + \eta_2(y_1)) > 0$$

for all values, y_1, satisfying the inequality in the definition (5.4). It follows from (5.5) and Lemma 3.1 that

$$\Pr(\eta_1(y_1) - a < Y_2 < \eta_2(y_1) - a) > \Pr(\eta_1(y_1) + a < Y_2 < \eta_2(y_1) + a)$$

for all values of y_1 appearing in (5.4). Consequently $\Pr(U_2(-a)) > \Pr(U_2(a))$ (unless $U_2(\pm a) = \emptyset$), and so $\Pr(S_2(-a)) > \Pr(S_2(a))$ (unless $S_2(\pm a) = \emptyset$). This concludes the proof of the lemma.

Observe that Lemmas 5.1 and 5.2, taken together, prove that



L is the line segment(s) $\{(y_1, y_2): y_1 = D \text{ (fixed)}\} \cap (U_2(\pm a))$

$$I = (D, \eta_1(D)-a),\ II = (D, \eta_2(D)-a),$$

$$III = (D, \eta_1(D)+a),\ IV = (D, \eta_2(D)+a)$$

(Note that the intersection of L with the y_2 axis divides L below its midpoint. This reflects (5.5).)

Fig. 5.2

Proposition 3 is valid when $p = 4$ since $Pr(S(\pm a)) = Pr(S_1(\pm a)) + Pr(S_2(\pm a))$. This completes the proof of Proposition 3 (and hence of Conjecture 1) for the case $p = 4$.

6. Proof of Proposition 3 (and Conjecture 1) for $p = 5$.

This proof has many elements of the proof of the preceding section for $p = 4$, but it also contains several steps which have no analog there. Without loss of generality assume $\rho_1 \geqq \rho_2 \geqq \rho_3$ throughout this section; as well as $p = 5$ and the remaining assumptions of Proposition 3.

Define W_i, $i = 1,2,3$, to be independent normal random variables with means θ_i and variances ρ_i^2. Let

$$T(a) = \{w \in \mathbb{R}^3\colon\ w - c \in S(a)\}$$

$$T_1(a) = \{w \in \mathbb{R}^3\colon\ w \in T(a),$$

$$|w_i - w_j| \leqq k(\rho_i^2 + \rho_j^2)^{1/2} - (c_i - c_j) \quad \text{for} \quad 1 \leqq i < j \leqq 3\}$$

$$S_1(a) = \{v \in \mathbb{R}^3\colon\ v + c \in T_1(a)\}$$

$$S_2(a) = S(a) - S_1(a).$$

Again, $Pr_{\theta = -c}(T_1(a)) = Pr(S_1(a))$; and the first half of the proof of Proposition 3 is to show

Lemma 6.1: If $\theta_i \leqq 0$, $i = 1,\ldots,p-2$ then $Pr_\theta(T_1(-a)) \geqq Pr_\theta(T_1(a))$, with equality if and only if $\theta = 0$. Consequently $Pr(S_1(-a)) > Pr(S_1(a))$.

Proof: The second sentence of the lemma follows from the first, as in the proof of Lemma 4.1.

Also as in the proof of Lemma 4.1, the sets $T_1(\pm a)$ are of the form $R^{\pm}$ of Lemma 4.1. Lemma 6.1 then follows directly from Lemma 4.1.

The second part of the proof of Proposition 3 is to show

<u>Lemma 6.2:</u> $\Pr(S_2(-a)) \geqq \Pr(S_2(a))$. (Again it can be shown that equality occurs if and only if $S_2(\pm a) = \emptyset$.)

<u>Proof:</u> The sets $S_2(\pm a)$ are each broken into three disjoint subsets. These are

$$Q_2(\pm a) = \{v \in S_2(\pm a): v_2 - v_3 \geqq (d_2 - d_3) + (c_2 - c_3)\}$$

$$Q_{3(1)}(\pm a) = \{v \in S_2(\pm a): v_2 - v_3 < (d_2 - d_3) + (c_2 - c_3), \; v_1 - v_2 \leqq -(d_1 - d_2) + (c_1 - c_2)\}$$

$$Q_{3(2)}(\pm a) = \{v \in S_2(\pm a): v_2 - v_3 < (d_2 - d_3) + (c_2 - c_3), \; v_1 - v_2 > -(d_1 - d_2) + (c_1 - c_2)\}$$

For the remainder of this proof we assume $\rho_{p-1} > \rho_3$; for otherwise $S_2(\pm a) = \emptyset$, and the lemma is trivially true. (See the appendix concerning Figure 2.1 to verify this assertion.)

(Geometrically these three sets can be visualized as follows: the sets $S_2(\pm a)$ are composed of line segments of the form $\{\nu + b\underline{1}: b \in \mathbb{R}\} \cap S_2(\pm a)$ for various $\nu \in \mathbb{R}_3$. $Q_2(\pm a)$ consists of those line segments which exit $S_2(\pm a)$ at $v_2 = d_2 + c_2 \pm a$. $Q_{3(i)}$ consists of those line segments which exit $S_2(\pm a)$ at $v_3 = d_3 + c_3 \pm a$ and enter at $v_i = -d_i + c_i \pm a$, $i = 1, 2$. (Thus, Q_2 could, more properly be labelled $Q_{2(1)}$.))

Consider the sets $Q_2(\pm a)$. Since $v_2 - v_3 \geqq 0$, it must be that either $v_1 - v_2 \leqq -k(\rho_1^2 + \rho_2^2)^{1/2} + 2(c_1 - c_2)$ or $v_1 - v_2 = v_1 - v_3 - (v_2 - v_3) \leqq -k(\rho_1^2 + \rho_3^2)^{1/2} + 2(c_1 - c_3)$ $-((d_2 - d_3) + (c_2 - c_3))$. It is shown in the appendix that the latter lower bound is larger. It follows that $v \in Q_2(\pm a)$ only if $v_1 - v_2 = \delta_{12}$ satisfies

$$(6.1) \qquad \max(-(d_1 + d_2) + (c_1 - c_2), -k(\rho_1^2 + \rho_2^2)^{1/2}) < \delta_{12} \leqq -k(\rho_1^2 + \rho_3^2)^{1/2} +2(c_1 - c_3) - (d_2 - d_3) - (c_2 - c_3).$$

Furthermore, it is shown in the appendix that (6.1) together with

$$(6.2) \qquad v_2 - k(\rho_2^2 + \rho_3^2)^{1/2} < v_3 < \min\left(v_2 + \delta_{12} + k(\rho_1^2 + \rho_3^2)^{1/2}, v_2 - (d_2 - d_3) - (c_2 - c_3)\right)$$

and

$$(6.3) \qquad -d_1 + c_1 \pm a < v_1 = v_2 + \delta_{12} < d_2 + c_2 + \delta_{12} \pm a$$

are necessary and sufficient for $v \in Q_2(\pm a)$.

As in the proof of Lemma 5.2 define $Y_1 = V_1 - V_2$ and $Y_2 = V_1/\rho_1^2 + V_2/\rho_2^2$. For fixed, given, $Y_1 = \delta_{12}$ satisfying (6.1) the regions $Q_2(\pm a)$ can be written because of (6.2) and (6.3) in terms of Y_2, V_3 as

$$(6.4)\qquad Q_2(\pm a) \cap \{Y_1 = \delta_{12}\}$$
$$= \{(y_2, v_3):\ \eta_1(\delta_{12}) \pm a < y_2 < \eta_2(\delta_{12}) \pm a\ ,\ \ \eta_3(\delta_{12}) + \eta_5\, y_2 < v_3 < \eta_4(\delta_{12}) + \eta_5\, y_2\}$$

where δ_{12} satisfies (6.1) and η_1, η_2 are as defined in (5.4) and η_3, η_4, η_5 are described more explicitly in the appendix. It is verified there that $\eta_5 > 0$,

$$(6.5)\qquad \eta_1(\delta_{12}) + \eta_2(\delta_{12}) \geqq 0\ ,$$

and

$$(6.6)\qquad (\eta_3(\delta_{12}) + \eta_4(\delta_{12})) + \eta_5(\eta_1(\delta_{12}) + \eta_2(\delta_{12})) \geqq 0$$

for all δ_{12} satisfying (6.1). It follows from Lemma 4.2 that $\Pr(Q_2(-a) \mid Y_1 = \delta_{12}) \geqq \Pr(Q_2(a) \mid Y_1 = \delta_{12})$ and hence

$$(6.7)\qquad \Pr(Q_2(-a)) \geqq \Pr(Q_2(a))$$

The sets $Q_{3(i)}$ are handled rather similarly to Q_2. For $v \in Q_{3(1)}(\pm a)$ the difference $v_1 - v_3 = \delta_{13}$ satisfies

$$\max\,(-(d_1 + d_3) + (c_1 - c_3),\ -k(\rho_1^2 + \rho_3^2)^{1/2})$$
$$(6.8)\qquad < \delta_{13} < -k(\rho_2^2 + \rho_3^2)^{1/2} - (d_1 - d_2) - (c_1 - c_2) + 2(c_1 - c_3).$$

See the appendix. It is also verified there that $v \in Q_{3(1)}(\pm a)$ if and only if (6.8) is satisfied along with and

$$(6.15) \qquad v_2 - (d_1 - d_2) + (c_1 - c_2) < v_1 < v_2 + k(\rho_1^2 + \rho_2^2)^{1/2}$$

and

$$(6.16) \qquad -d_2 + c_2 \pm a < v_2 = v_3 + \delta_{23} < d_3 + c_3 + \delta_{23} \pm a.$$

As in the previous two cases Lemma 4.2 may be applied to yield

$$(6.17) \qquad \Pr(Q_{3(2)}(-a)) > \Pr(Q_{3(2)}(a)).$$

Combining (6.7), (6.13) and (6.17) yields the desired result, that

$$\Pr(S_2(-a)) \geqq \Pr(S_2(+a)).$$

This completes the proof of Lemma 6.2.

Lemma 6.1 and 6.2 combined show that Conjecture 1 is valid for the case $p = 5$. This completes the proof of Theorem 1.

$$(6.9) \qquad v_1 + (d_1 - d_2) - (c_1 - c_2) \leqq v_2$$

$$< v_3$$

$$+ \delta_{13} + k(\rho_1^2 + \rho_2^2)^{1/2}$$

and

$$(6.10) \qquad -d_1 + c_1 \pm a < v_1 = v_3 + \delta_{13} < d_3 + c_3 + \delta_{13} \pm a.$$

Hence if we define $Y_3 = V_1 - V_3$ and $Y_4 = V_1/\rho_1^2 + V_3/\rho_3^2$ then the regions $Q_{3(1)}(\pm a)$ can be written in terms of the values of Y_3, Y_4, V_2 as (except for a set of measure zero)

$$Q_{3(1)}(\pm a) = \{(y_3, y_4, v_2)\colon y_3 = \delta_{13},\ \delta_{13} \text{ satisfies } (6.8),$$

$$\eta_6(\delta_{13}) \pm a < y_4 < \eta_7(\delta_{13}) \pm a,$$

$$\eta_8(\delta_{13}) + \eta_{10}\, y_4 < v_2 < \eta_9(\delta_{13}) + \eta_{10}\, y_4\}.$$

It is checked in the appendix that $\eta_{10} > 0$,

$$(6.11) \qquad \eta_6(\delta_{13}) + \eta_7(\delta_{13}) \geqq 0$$

and

$$(6.12) \qquad \eta_8(\delta_{13}) + \eta_9(\delta_{13}) + \eta_{10}(\eta_6(\delta_{13}) + \eta_7(\delta_{13})) \geqq 0.$$

It follows from Lemma 4.2 that

$$(6.13) \qquad \Pr(Q_{3(1)}(-a)) \geqq \Pr(Q_{3(1)}(a)).$$

For $v \in Q_{3(2)}(\pm a)$ the defining inequalities are

$$(6.14) \qquad \max(-k(\rho_2^2 + \rho_3^2)^{1/2},\ -(d_2 + d_3) + (c_2 - c_3)$$

$$< \delta_{23} = v_2 - v_3 < -k(\rho_2^2 + \rho_3^2)^{1/2} + 2(c_2 - c_3)$$

$$\Gamma_{p-1}(\rho) = \lim_{\Delta\to 0} \Delta^{-1}[\Pr(|\beta^*_{p-1,p}(\rho,\Delta)\cdot Z|$$

$$\geq k,\ \max\{|\beta^*_{i,j}(\rho,\Delta)\cdot Z| : (i,j) \neq (p-1,p)\} < k)$$

$$+\Pr(\max\{|\beta^*_{i,j}(\rho,\Delta)\cdot Z| : (i,j) \neq (p-1,p)\} \geq k)$$

$$-\Pr(|\beta_{p-1,p}(\rho)\cdot Z| \geq k,$$

$$\max\{|\beta_{i,j}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\} < k)$$

$$-\Pr(\max\{|\beta_{i,j}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\} \geq k)]$$

$$= 2 \lim_{\Delta\to 0} \Delta^{-1}[\Pr(\beta^*_{p-1,p}(\rho,\Delta)\cdot Z \geq k,$$

$$\beta_{p-1,p}(\rho)\cdot Z < k;\ \max\{|\beta_{ij}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\} < k)$$

$$-\Pr(\beta^*_{p-1,p}(\rho,\Delta) < k, \beta_{p-1,p}(\rho)\cdot Z$$

$$\geq k; \max\{|\beta_{ij}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\} < k)]$$

$$= 2 \lim_{\Delta\to 0} \Delta^{-1}[E(\Pr(k-\Delta\rho_{p-1}v_{p-1}/(\rho_p^2+\rho_{p-1}^2) \leq V_p < k,$$

$$\max\{|\beta_{ij}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\} < k \mid V_{p-1} = v_{p-1}))$$

$$-E(\Pr(k-\Delta\rho_{p-1}v_{p-1}/(\rho_p^2+\rho_{p-1}^2) > V_p \geq k,$$

$$\max\{|\beta_{ij}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\} < k \mid V_{p-1} = v_{p-1}))]$$

$$= 2\varphi(k)[E((\rho_{p-1}V_{p-1}/(\rho_p^2+\rho_{p-1}^2))^+$$

$$\Pr(\max\{|\beta_{ij}(\rho)\cdot Z| : (i,j) \neq (p-1,p)\}$$

Appendix (Proofs, computations, and a conjecture)

Proposition 1 implies Conjecture 1: Let $\rho_i^2 = N_i^{-1}$. Then $\rho_i Z_i$ has the same distribution as $X_{i.}-\mu_i$. Let $k = sq_{p,\nu}^{(\alpha)}/2^{1/2}$. Then since S is independent of $\{X_i, i = 1,\ldots,p\}$ (and of $\{Z_i\}$ as well) (1.3) yields

$$\Pr\{|(X_{i.}-X_{j.})-(\mu_i-\mu_j)| \geq s(N_i^{-1}+N_j^{-1})^{1/2}q_{p,\nu}^{(\alpha)}/2^{1/2}$$

$$\text{for some } i \neq j\} \leq \Pr\{|Z_i-Z_j| \geq sq_{p,\nu}^{(\alpha)} \text{ for some } i \neq j\} = \alpha$$

by the definition of $q_{p,\nu}^{(\alpha)}$. ||

Calculations for (2.4):

$$\beta_{p-1,p}(\rho+\Delta e_p)\cdot Z = (-(\rho_p+\Delta)Z_p+\rho_{p-1}Z_{p-1})/((\rho_p+\Delta)^2+\rho_{p-1}^2)^{1/2}$$

$$= (-\rho_p Z_p+\rho_{p-1}Z_{p-1}-\Delta Z_p)(1-\Delta\rho_p/(\rho_p^2+\rho_{p-1}^2)+o(\Delta))/(\rho_p^2+\rho_{p-1}^2)^{1/2}$$

$$= V_p-\Delta Z_p/(\rho_p^2+\rho_{p-1}^2)^{1/2}-\Delta\rho_p V_p/(\rho_p^2+\rho_{p-1}^2)+o_p(\Delta)$$

$$= V_p+\Delta\rho_{p-1}V_{p-1}/(\rho_p^2+\rho_{p-1}^2)+o_p(\Delta),$$

since

$$Z_{p-1} = (-\rho_p V_{p-1}+\rho_{p-1}V_p)/(\rho_p^2+\rho_{p-1}^2)^{1/2}$$

$$Z_p = -(\rho_{p-1}V_{p-1}+\rho_p V_p)/(\rho_p^2+\rho_{p-1}^2)^{1/2}. \tag{A1}$$

Calculation for (2.5): Continuing after the first step of (2.5), we get

$$< k|V_p = k, V_{p-1}))+E((\rho_{p-1}V_{p-1}/(\rho_p^2+\rho_{p-1}^2))^-$$

$$\Pr(\max\{|\beta_{ij}(\rho)\cdot Z|:(i,j)\neq(p-1,p)\}< k|V_p = k, V_{p-1}))]$$

$$= 2\varphi(k)E((\rho_{p-1}V_{p-1}/(\rho_p^2+\rho_{p-1}^2))^+(\hat{S}\ (V_{p-1})-\hat{S}\ (-V_{p-1})))$$

where $a^+ = \max(a,0)$, $a^- = \min(a,0)$.

<u>Computations for (2.6)</u>: In view of (A1),

$$\hat{S}(v) = \{(v_1,\ldots,v_{p-2}):|v_i-v_j| < k(\rho_i^2+\rho_j^2)^{1/2}, |\rho_p z_p+v_i|$$

$$< k(\rho_p^2+\rho_i^2)^{1/2}, |\rho_{p-1}z_{p-1}+v_i| < k(\rho_{p-1}^2+\rho_i^2)^{1/2}\}$$

$$= \{(v_1,\ldots,v_{p-2}):|v_i-v_j| < k(\rho_i^2+\rho_j^2)^{1/2},$$

$$|-k\rho_p^2/(\rho_p^2+\rho_{p-1}^2)^{1/2} - \rho_p\rho_{p-1}v/(\rho_p^2+\rho_{p-1}^2)^{1/2}+v_i|$$

$$< k(\rho_p^2+\rho_i^2)^{1/2}, |+k\rho_{p-1}^2/(\rho_p^2+\rho_{p-1}^2)^{1/2}$$

$$-\rho_p\rho_{p-1}v/(\rho_p^2+\rho_{p-1}^2)^{1/2}+v_i| < k(\rho_{p-1}^2+\rho_i^2)^{1/2}\}$$

$$= \{(v_1,\ldots,v_{p-2}):|v_i-v_j| < k(\rho_i^2+\rho_j^2)^{1/2},$$

$$a+k\rho_p^2/(\rho_p^2+\rho_{p-1}^2)^{1/2}-k(\rho_p^2+\rho_i^2)^{1/2} < v_i$$

$$< a-k\rho_{p-1}^2/(\rho_p^2+\rho_{p-1}^2)^{1/2}+k(\rho_{p-1}^2+\rho_i^2)^{1/2}\}$$

where $a = \rho_p\rho_{p-1}v/(\rho_p^2+\rho_{p-1}^2)^{1/2}$ since

$$\frac{k\rho_p^2}{(\rho_p^2+\rho_{p-1}^2)^{1/2}} - k(\rho_p^2+\rho_i^2)^{1/2} > -\frac{k\rho_{p-1}^2}{(\rho_p^2+\rho_{p-1}^2)^{1/2}} - k(\rho_{p-1}^2+\rho_i^2)^{1/2}$$

(because $(\rho_p^2+\rho_{p-1}^2)^{1/2}+(\rho_{p-1}^2+\rho_i^2)^{1/2} > (\rho_p^2+\rho_i^2)^{1/2}$, and

$$-\frac{k\rho_{p-1}^2}{(\rho_p^2+\rho_{p-1}^2)^{1/2}} + k(\rho_{p-1}^2+\rho_p^2)^{1/2} < \frac{k\rho_p^2}{(\rho_p^2+\rho_{p-1}^2)^{1/2}} + k(\rho_p^2+\rho_i^2)^{1/2} .$$

Defining c_i, d_i as in (2.6) now yields the desired expression.

Which vertices are in $\overline{S}(\pm a)$? (See Figure 2.1):

The vertex $P_1 = (a+d_1+c_1, a+d_2+c_2)$. Now

(A2) $$0 \leqq (d_1+c_1)-(d_2+c_2) = k((\rho_{p-1}^2+\rho_1^2)^{1/2}-(\rho_{p-1}^2+\rho_2^2)^{1/2})$$
$$\leqq k(\rho_1^2+\rho_2^2)^{1/2} .$$

Hence $P_1 \in \overline{S}(a)$.

Similarly $P_3 = (a-d_1+c_1, a-d_2+c_2)$ so that

$$0 \geqq (-d_1+c_1)-(-d_2+c_2) = -k((\rho_p^2+\rho_1^2)^{1/2}-(\rho_p^2+\rho_2^2)^{1/2})$$
$$\geqq -k(\rho_1^2+\rho_2^2)^{1/2} .$$

Hence $P_3 \in \overline{S}(a)$.

The vertex $P_2 = (a+d_1+c_1, a-d_2+c_2)$. Now,

(A3) $$0 \leqq d_1+c_1-(-d_2+c_2) = k((\rho_{p-1}^2+\rho_1^2)^{1/2}+(\rho_p^2+\rho_2^2)^{1/2}-(\rho_p^2+\rho_{p-1}^2)^{1/2})$$
$$\leqq k(\rho_1^2+\rho_2^2)^{1/2}$$

if and only if $\rho_{p-1}^2 \leqq \rho_2^2$ by application of the Cauchy-Schwartz inequality. (We have assumed $\rho_1 \geqq \rho_2$.) Hence $P_2 \in \overline{S}(a)$ if and only if $\rho_{p-1}^2 \leqq \rho_2^2$. Similarly $P_4 \in \overline{S}(a)$ if and only if $\rho_{p-1}^2 \leqq \rho_1^2$.

Verification of (5.5) in the proof of Lemma 5.2: According to Figure 5.2, $\eta_1(y_1) = (-d_1+c_1)/\rho_1^2+(-d_1+c_1-y_1)/\rho_2^2$ and $\eta_2(y_1) = (d_2+c_2+y_1)/\rho_1^2+(d_2+c_2)/\rho_2^2$. Hence, we need to prove

$$\text{(A4)}\qquad 0 \overset{?}{<} \eta_1(y_1)+\eta_2(y_1) = (1/\rho_1^2+1/\rho_2^2)(-d_1+d_2+c_1+c_2)-y_1(1/\rho_2^2-1/\rho_1^2)$$

for all y_1 satisfying the inequality in (5.4). (The symbol $\overset{?}{<}$ denotes an inequality to be verified, etc.) Taking (5.4) into account, (A4) will be valid if

$$\text{(A5)}\qquad (k(\rho_1^2+\rho_2^2)^{1/2}-2(c_1-c_2))(\rho_1^2-\rho_2^2) \overset{?}{>} (\rho_1^2+\rho_2^2)(d_1-d_2-c_1-c_2).$$

Since

$$2(c_1-c_2)(\rho_1^2-\rho_2^2)-(c_1+c_2)(\rho_1^2+\rho_2^2) \leqq (c_1-c_2)(\rho_1^2-\rho_2^2)$$

$$\leqq (c_1-c_2)(\rho_1^2+\rho_2^2)$$

it suffices to verify

$$\text{(A6)}\qquad k(\rho_1^2+\rho_2^2)^{1/2}(\rho_1^2-\rho_2^2) \overset{?}{>} (\rho_1^2+\rho_2^2)(d_1-d_2+c_1-c_2)$$

$$= k(\rho_1^2+\rho_2^2)((\rho_{p-1}^2+\rho_1^2)^{1/2}-(\rho_{p-1}^2+\rho_2^2)^{1/2}).$$

Now, $(\rho_1^2+\rho_2^2)^{1/2}(\rho_1^2-\rho_2^2) = (\rho_1^2+\rho_2^2)^{1/2}(\rho_1+\rho_2)(\rho_1-\rho_2) > (\rho_1^2+\rho_2^2)(\rho_1-\rho_2)$. Hence it suffices to verify

$$\text{(A7)} \qquad (\rho_1^2+\rho_2^2)(\rho_1-\rho_2) \geq (\rho_1^2+\rho_2^2)$$

$$((\rho_{p-1}^2+\rho_1^2)^{1/2}-(\rho_{p-1}^2+\rho_2^2)^{1/2}).$$

Now $(m^2+\rho_1^2)^{1/2}-(m^2+\rho_2^2)^{1/2}$ is a decreasing function of $m \geq 0$. Hence (A7) is a valid inequality. This verifies (A4), and consequently (5.5).

Verification of (6.1): It is required to show

$$-k(\rho_1^2+\rho_3^2)^{1/2}+2(c_1-c_3)-(d_2-d_3)$$

$$-(c_2-c_3) \geq -k(\rho_1^2+\rho_2^2)^{1/2}+2(c_1-c_2).$$

This is equivalent to

$$\text{(A.8)} \qquad k(\rho_1^2+\rho_2^2)^{1/2} \geq k(\rho_1^2+\rho_3^2)^{1/2}+(d_2-d_3)-(c_2-c_3)$$

$$= k(\rho_1^2+\rho_3^2)^{1/2}+k(\rho_p^2+\rho_2^2)^{1/2}-k(\rho_p^2+\rho_3^2)^{1/2} .$$

Since $\rho_3 \leq \rho_2 \leq \rho_1 \leq \rho_p$ this last inequality is valid by application of the Cauchy-Schwartz inequality.

Verification of (6.2) and (6.3): The conditions stated in (6.2) and (6.3) are obviously necessary for $v \in Q_2(\pm a)$. One further defining condition for $Q_2(\pm a)$ is that $v_3 > -d_3+c_3$. But note that the lower bound already stated by (6.2) and (6.3) on v_3 is

$$\text{(A.9)} \quad v_3 > v_1-\delta_{12}-k(\rho_2^2+\rho_3^2)^{1/2} \geq -d_1+c_1+k(\rho_1^2+\rho_3^2)^{1/2}-2(c_1-c_3)$$

$$+d_2-d_3+c_2-c_3-k(\rho_2^2+\rho_3^2)^{1/2} = k(\rho_1^2+\rho_3^2)^{1/2}-k(\rho_2^2+\rho_3^2)^{1/2}$$

$$-(d_1-d_2)-(c_1-c_2)-d_3+c_3$$

$$= k(\rho_1^2+\rho_3^2)^{1/2}-k(\rho_2^2+\rho_3^2)^{1/2}-k(\rho_{p-1}^2+\rho_1^2)^{1/2}$$

$$+k(\rho_{p-1}^2+\rho_2^2)^{1/2}-d_3+c_3$$

$$\geqq -d_3+c_3$$

by the Cauchy-Schwartz inequality since $\rho_{p-1} > \rho_3$. It follows that the conditions stated in (6.2) and (6.3) already imply the condition $v_3 > -d_3+c_3$. An enumeration of the conditions for $v \in Q_2(\pm a)$ now shows that (6.1)-(6.3) are sufficient for $v \in Q_2(\pm a)$.

Verification of (6.5): It follows from (6.3) as in (A4), that

$$\eta_1(\delta_{12})+\eta_2(\delta_{12}) = (1/\rho_1^2+1/\rho_2^2)(-d_1+d_2+c_1+c_2)-\delta_{12}(1/\rho_2^2-1/\rho_1^2),$$

but here δ_{12} is subject only to the upper bound in (6.1). Thus it is required to verify

$$\text{(A10)} \qquad (\rho_1^2-\rho_2^2)(+k(\rho_1^2+\rho_3^2)^{1/2}+d_2-d_3+(c_2-c_3)-2(c_1-c_3))$$

$$\geqq (\rho_1^2+\rho_2^2)(d_1-d_2-(c_1+c_2)).$$

The left hand side is equal to

$$k(\rho_1^2-\rho_2^2)((\rho_1^2+\rho_3^2)^{1/2}-(\rho_p^2+\rho_3^2)^{1/2}+(\rho_{p-1}^2+\rho_2^2)^{1/2}-(\rho_{p-1}^2+\rho_1^2)^{1/2}$$

$$+(\rho_p^2+\rho_1^2)^{1/2})$$

$$> k(\rho_1^2-\rho_2^2)(\rho_2+(\rho_p^2+\rho_1^2)^{1/2}-\rho_p)$$

since $(\rho_1^2+\rho_3^2)^{1/2}-(\rho_p^2+\rho_3^2)^{1/2}>\rho_1-\rho_p$ and $(\rho_{p-1}^2+\rho_2^2)^{1/2}-(\rho_{p-1}^2+\rho_1^2)^{1/2}>\rho_2-\rho_1$.

At the same time the right hand side of (A 10) satisfies

$$\begin{aligned}&(\rho_1^2+\rho_2^2)(d_1-d_2-(c_1+c_2))\\&\quad\leq(\rho_1^2+\rho_2^2)(d_1-d_2-(c_1-c_2))\\&\quad=k(\rho_1^2+\rho_2^2)((\rho_p^2+\rho_1^2)^{1/2}-(\rho_p^2+\rho_2^2)^{1/2}).\end{aligned}$$

Consequently it suffices to show

$$(\rho_1^2-\rho_2^2)(\rho_2+\alpha(\rho_1))\geq(\rho_1^2+\rho_2^2)(\alpha(\rho_1)-\alpha(\rho_2))\tag{A.11}$$

where $\alpha(\rho)=(\rho_p^2+\rho^2)^{1/2}-\rho_p$. This can be rewritten as

$$\rho_1^2(\rho_2+\alpha(\rho_2))-2\rho_2^2\alpha(\rho_1)\geq\rho_2^2(\rho_2-\alpha(\rho_2))\tag{A.12}$$

Note that the right side of this expression is independent of ρ_1^2 whereas the left side satisfies

$$\begin{aligned}&\frac{\partial}{\partial(\rho_1^2)}[\rho_1^2(\rho_2+\alpha(\rho_2))-2\rho_2^2\alpha(\rho_1)]\\&\quad=(\rho_2+\alpha(\rho_2))-\rho_2^2/(\rho_p^2+\rho_1^2)^{1/2}\\&\quad\geq\rho_2(1-\rho_2/(\rho_p^2+\rho_1^2)^{1/2})>0.\end{aligned}\tag{A.13}$$

Thus the left side of (A.12) is increasing as a function of ρ_1^2, for fixed ρ_2, ρ_p, and so reaches its maximum when $\rho_2=\rho_p$ (since $\rho_1\leq\rho_p$).

Assume $\rho_1=\rho_p$. Then the left side of (A.11) becomes

$$\begin{aligned}(\rho_1^2-\rho_2^2)(\rho_2+\alpha(\rho_1))&\geq(\rho_p^2+\rho_2^2)^{1/2}(\rho_p-\rho_2)(\rho_2+(\sqrt{2}-1)\rho_p)\\&=(\rho_p^2+\rho_2^2)^{1/2}(\rho_p\rho_2(2-\sqrt{2})+\sqrt{2}\rho_p^2-\rho_p^2-\rho_2^2)\end{aligned}$$

Now since $\rho_p \geq \rho_2$,

$$\rho_2(2-\sqrt{2})+\sqrt{2}\rho_p = (\rho_2^2(2-\sqrt{2})^2+4\rho_2\rho_p(\sqrt{2}-1)+2\rho_p^2)^{1/2}$$

$$\geq (\rho_2^2((2-\sqrt{2})^2+4(\sqrt{2}-1))+2\rho_p^2)^{1/2}$$

$$= \sqrt{2}(\rho_2^2+\rho_p^2)^{1/2}.$$

Hence the left side of (A.11) now becomes

$$(\rho_1^2-\rho_2^2)(\rho_2+\alpha(\rho_1)) \geq (\rho_p^2+\rho_2^2)^{1/2}(\sqrt{2}\rho_p(\rho_2^2+\rho_p^2)^{1/2}$$

$$-\rho_p^2-\rho_2^2) = (\rho_p^2+\rho_2^2)(\sqrt{2}\rho_p-(\rho_p^2+\rho_2^2)^{1/2})$$

$$= (\rho_p^2+\rho_2^2)(\alpha(\rho_1)-\alpha(\rho_2))$$

this verifies that (A.11) is valid when $\rho_1 = \rho_p$; and hence by the reasoning at (A.13) verifes that (6.5) is valid for $\rho_2 \leq \rho_1 < \rho_p$.

<u>Calculation of</u> η_3, η_4, η_5 <u>and verification of (6.6)</u>: We do not need explicit expressions for η_3, η_4, η_5. These expressions could, however, be directly derived by substituting $v_2 = (\rho_1^2\rho_2^2y_2+\rho_2^2\delta_{12})/(\rho_1^2-\rho_2^2)$ in (6.2). Note that $\eta_5 = \rho_1^2\rho_2^2/(\rho_1^2-\rho_2^2) > 0$.

For verification of (6.6) it suffices to note that $\eta_3 + \eta_5\eta_1 - a = \inf\{v_3\colon v_3$ satisfies (6.2) and (6.3) for $Q_2(-a)\}$ and $\eta_4 + \eta_5\eta_2 + a - \sup\{v_3\colon v_3$ satisfies (6.2) and (6.3) for $Q_2(+a)\}$. Consequently

$$\eta_3 + \eta_5\eta_1-a=-d_1+c_1-\delta_{12}-k(\rho_2^2+\rho_3^2)^{1/2}-a$$

and

$$\eta_4 + \eta_5\eta_2+a=a+\min(d_2+c_2+\delta_{12}+k(\rho_1^2+\rho_3^2)^{1/2},$$

$$d_2+c_2-(d_2-d_3)-(c_2-c_3))$$

It can be calculated that in any case

$$\eta_4 + \eta_5\eta_2+a \geqq d_2+c_2+\delta_{12}+k(\rho_1^2+\rho_3^2)^{1/2}-2(c_1-c_3)+a$$

because of the lower bound, $\delta_{12} > -(d_1+d_2)+(c_1-c_2)$, in (6.1) and (A2) (with ρ_3 substituted for ρ_2, etc.). Hence

$$(\eta_3+\eta_5\eta_1-a)+(\eta_4+\eta_5\eta_2+a) \geqq -d_1+c_1-k(\rho_2^2+\rho_3^2)^{1/2}+d_2+c_2$$
$$+ k(\rho_1^2+\rho_3^2)^{1/2}-2(c_1-c_3) \geqq k(\rho_1^2+\rho_3^2)^{1/2}-k(\rho_2^2+\rho_3^2)^{1/2}$$
$$-(d_1-d_2)-(c_1-c_2) > 0$$

by (A.9). This verifies (6.6).

Verification of (6.8): If $v \in S_2(\pm a)$ then v must satisfy one of the inequalities

$$\text{(A.14)} \qquad \begin{aligned} v_1-v_3 &\leqq -k(\rho_1^2+\rho_3^2)^{1/2}+2(c_1-c_3) \\ v_1-v_2 &\leqq -k(\rho_1^2+\rho_2^2)^{1/2}+2(c_1-c_2) \\ v_2-v_3 &\leqq -k(\rho_2^2+\rho_3^2)^{1/2}+2(c_2-c_3) \end{aligned}$$

Suppose the second. Then, by the defining condition of $Q_{3(1)}$,

$$\begin{aligned} v_1-v_3 = v_1-v_2+v_2-v_3 &\leqq \\ &- k(\rho_1^2+\rho_2^2)^{1/2}+2(c_1-c_2)+d_2-d_3+c_2-c_3 \\ &= - k(\rho_1^2+\rho_2^2)^{1/2}+(d_2-d_3)-(c_2-c_3)+2(c_1-c_3) \\ &< -k(\rho_1^2+\rho_3^2)^{1/2}+2(c_1-c_3) \end{aligned}$$

as in (A.8). Hence the second inequality implies the first. Suppose the third. Then

$$v_1-v_3 = v_1-v_2+v_2-v_3 \leqq -(d_1-d_2)+(c_1-c_2)-k(\rho_2^2+\rho_3^2)^{1/2}$$
$$+ 2(c_2-c_3) = -k(\rho_2^2+\rho_3^2)^{1/2}-(d_1-d_2)-(c_1-c_2)+2(c_1-c_3).$$

Hence the third inequality directly implies (6.8). Finally, the first inequality implies (6.8) since

$$-k(\rho_1^2 + \rho_3^2)^{1/2} \leqq -k(\rho_2^2 + \rho_3^2)^{1/2}$$

$$-k(\rho_{p-1}^2 + \rho_1^2)^{1/2} + k(\rho_{p-1}^2 + \rho_2^2)^{1/2}$$

$$= -k(\rho_2^2 + \rho_3^2)^{1/2} - (d_1 - d_2) - (c_1 - c_2)$$

by (A.9).

Verification of (6.9): The defining property of $Q_{3(1)}$ includes the condition

$$v_2 < v_3 + (d_2 - d_3) + (c_2 - c_3). \tag{A.15}$$

However,

$$\begin{aligned}
&\delta_{13} + k(\rho_1^2 + \rho_2^2)^{1/2} < -k(\rho_2^2 + \rho_3^2)^{1/2} \\
&-(d_1 - d_2) - (c_1 - c_2) + 2(c_1 - c_3) \\
&+ k(\rho_1^2 + \rho_2^2)^{1/2} \\
&= -k(\rho_2^2 + \rho_3^2)^{1/2} + k(\rho_1^2 + \rho_2^2)^{1/2} \\
&-(d_1 - d_3) + (c_1 - c_3) \\
&+(d_2 - d_3) + (c_2 - c_3) \\
&< (d_2 - d_3) + (c_2 - c_3).
\end{aligned}$$

Hence the condition stated in (6.9) already implies (A.15). The remaining assertions implicit in (6.8)-(6.10) can be directly checked from the definition of $Q_{3(1)}$.

Verification of (6.11): The constants η_6 and η_7 are the same as η_1 and η_2 with the roles of ρ_2 and ρ_3 interchanged. Hence we need to verify

$$\eta_6(\delta_{13}) + \eta_7(\delta_{13})$$
$$= (1/\rho_1^2 + 1/\rho_3^2)(-d_1 + d_3 + c_1 + c_3 - \delta_{13}(1/\rho_3^2 - 1/\rho_1^2)) \geqq 0,$$

for all δ_{13} satisfying (6.8). This is equivalent to

$$(\rho_1^2 - \rho_3^2)(k(\rho_2^2 + \rho_3^2)^{1/2} + (d_1 - d_2)$$
$$+ (c_1 - c_2) - 2(c_1 - c_3)) \geqq$$
$$(\rho_1^2 + \rho_3^2)(d_1 - d_3 - c_1 - c_3).$$

Note that the right side is independent of ρ_2 and

$$k(\rho_2^2 + \rho_3^2)^{1/2} + (d_1 - d_2) + (c_1 - c_2)$$
$$= k[(\rho_2^2 + \rho_3^2)^{1/2} + (\rho_{p-1}^2 + \rho_1^2)^{1/2} - (\rho_{p-1}^2 + \rho_2^2)^{1/2}]$$
$$\leqq k[(2\rho_3^2)^{1/2} + (\rho_{p-1}^2 + \rho_1^2)^{1/2} - (\rho_{p-1}^2 + \rho_3^2)^{1/2}].$$

Hence it suffices to show

$$(\rho_1^2 - \rho_3^2)(k(2\rho_3^2)^{1/2} + k(\rho_{p-1}^2 + \rho_1^2)^{1/2}$$
$$-k(\rho_{p-1}^2 + \rho_3^2)^{1/2}) - 2(c_1 - c_3))$$
$$= k(\rho_1^2 - \rho_3^2)((2\rho_3^2)^{1/2} + (\rho_p^2 + \rho_1^2)^{1/2}$$
$$- (\rho_p^2 + \rho_3^2)^{1/2})$$
$$\geqq k(\rho_1^2 + \rho_3^2)((\rho_p^2 + \rho_1^2)^{1/2} - (\rho_p^2 + \rho_3^2)^{1/2})$$

Now,

$$((\rho_p^2 + \rho_3^2)^{1/2} - (2\rho_3^2)^{1/2})^2$$

$$= (\rho_p^2 + \rho_3^2) - 2(2\rho_3^2)^{1/2}(\rho_p^2 + \rho_3^2)^{1/2} + 2\rho_3^2$$

$$= (\rho_p - \rho_3)^2 + 2\rho_3\rho_p + 2\rho_3^2$$

$$-2\sqrt{2}\,\rho_3(\rho_p^2 + \rho_3^2)^{1/2}$$

$$< (\rho_p - \rho_3)^2$$

since $\quad (\rho_p + \rho_3) - \sqrt{2}(\rho_p^2 + \rho_3^2)^{1/2} < 0.$

In view of this it suffices to show

$$(\rho_1^2 - \rho_3^2)(\rho_3 + (\rho_p^2 + \rho_1^2)^{1/2} - \rho_p)$$

$$\geqq (\rho_1^2 + \rho_3^2)((\rho_p^2 + \rho_1^2)^{1/2} - (\rho_p^2 + \rho_3^2)^{1/2})$$

this expression is the same as (A.11) with ρ_3 substituted for ρ_2. Hence this expression is a valid inequality and the verification of (6.11) is complete.

Verification of (6.12): Reasoning as in the verification of (6.6) we have that $\eta_{10} > 0$ and

$$\eta_8 + \eta_9 + \eta_{10}(\eta_6 + \eta_7) = -d_1 + c_1 + d_1 - d_2$$

$$-(c_1 - c_2)) + (d_3 + c_3) + \delta_{13}$$

$$+ k(\rho_1^2 + \rho_2^2)^{1/2} > \max(-(d_1 + d_2)$$

$$+ (c_1 + c_2) + k(\rho_1^2 + \rho_2^2)^{1/2}, -(d_2 - d_3)$$

$$+ c_2 + c_3 - k(\rho_1^2 + \rho_3^2)^{1/2} + k(\rho_1^2 + \rho_2^2)^{1/2})$$

Now,

$$k(\rho_1^2 + \rho_2^2)^{1/2} - k(\rho_1^2 + \rho_3^2)^{1/2} - (d_2 - d_3)$$

$$+ (c_2 + c_3) > k(\rho_1^2 + \rho_2^2)^{1/2} - k(\rho_1^2 + \rho_3^2)^{1/2}$$

$$- k(\rho_p^2 + \rho_2^2)^{1/2} + k(\rho_p^2 + \rho_3^2)^{1/2} > 0.$$

This verifies (6.12).

<u>Verification of (6.14):</u> Since

$$v_1 - v_2 > -(d_1 - d_2) + (c_1 - c_2) = -(d_1 - d_2)$$

$$-(c_1 - c_2) + 2(c_1 - c_2) = -k(\rho_{p-1}^2 + \rho_1^2)^{1/2} + k(\rho_{p-1}^2 + \rho_2^2)^{1/2}$$

$$+ 2(c_1 - c_2) > -k(\rho_1^2 + \rho_2^2)^{1/2} + 2(c_1 - c_2)$$

if follows that either

$$v_2 - v_3 \leqq -k(\rho_2^2 + \rho_3^2)^{1/2} + 2(c_2 - c_3)$$

or

$$v_2 - v_3 = v_2 - v_1 + v_1 - v_3$$

$$\leqq d_1 - d_2 - (c_1 - c_2) - k(\rho_1^2 + \rho_3^2)^{1/2}$$

$$+ 2(c_1 - c_3)$$

$$= -k(\rho_1^2 + \rho_3^2)^{1/2} + (d_1 - d_2) + (c_1 - c_2)$$

$$+ 2(c_2 - c_3)$$

$$= -k(\rho_1^2 + \rho_3^2)^{1/2} + k(\rho_{p-1}^2 + \rho_1^2)^{1/2}$$

$$-k(\rho_{p-1}^2 + \rho_2^2)^{1/2} + 2(c_2 - c_3)$$

$$< -k(\rho_2^2 + \rho_3^2)^{1/2} + 2(c_2 - c_3)$$

Hence the former inequality concerning $v_2 - v_3$ is the proper expression for (6.14).

Verification for (6.15): It is necessary to check that (6.15) implies $v_1 - v_3 \dot{>} -k(\rho_1^2 + \rho_3^2)^{1/2}$. Now, (6.15) yields

$$v_1 - v_3 = v_1 - v_2 + v_2 - v_3 > -(d_1 - d_2)$$
$$+ (c_1 - c_2) - k(\rho_2^2 + \rho_3^2)^{1/2}$$
$$= -k(\rho_p^2 + \rho_1^2)^{1/2} + k(\rho_p^2 + \rho_2^2)^{1/2} - k(\rho_2^2 + \rho_3^2)^{1/2}$$
$$> -k(\rho_1^2 + \rho_3^2)^{1/2}$$

which verifies the desired property.

Verification for (6.17):

It is necessary to verify the analogs of properties (6.5) and (6.6). The first of these properties will be verified if

$$(1/\rho_2^2 + 1/\rho_3^2)(-d_2 + d_3 + c_2 + c_3)$$
$$-\delta_{23}(1/\rho_3^2 - 1/\rho_2^2) \dot{>} 0.$$

where δ_{23} satisfies (6.14). Note that this expression is exactly (A4) with (ρ_2, ρ_3) substituted for (ρ_1, ρ_2). Hence this expression is valid.

The analog of (6.6) requires

$$-d_2 + c_2 - (d_1 - d_2) + (c_1 - c_2)$$
$$+ d_3 + c_3 - k(\rho_2^2 + \rho_3^2)^{1/2} + 2(c_2 - c_3)$$
$$+ k(\rho_1^2 + \rho_2^2)^{1/2} \dot{>} 0$$

The left side of this expression is greater than

$$-d_1 + d_3 + c_1 - c_3 + k(\rho_1^2 + \rho_2^2)^{1/2}$$

$$-k(\rho_2^2 + \rho_3^2)^{1/2} = -k(\rho_p^2 + \rho_1^2)^{1/2}$$

$$+ k(\rho_p^2 + \rho_3^2)^{1/2} + k(\rho_1^2 + \rho_2^2)^{1/2} - k(\rho_2^2 + \rho_3^2)^{1/2}$$

$$> 0.$$

This verifies the desired property.

Postscript (A conjecture): The method of proof used in Sections 5, 6 is not-in principle-limited to the cases $p = 4,5$. Take, for example, the case $p = 6$ and assume $\rho_1 \geqq \rho_2 \geqq \rho_3 \geqq \rho_4$. One can still define sets $S_1(\pm a)$ and $S_2(\pm a)$ analogous to those in Section 6. Lemma 4.1 still yields $\Pr(S_1(-a)) \geqq \Pr(S_1(a))$. The sets $S_2(\pm a)$ can be broken into several disjoint pieces in analogy with the sets $Q_{j(i)}$ of Section 6. The first of these sets - Q_2 (or $Q_{2(1)}$) - could be written as

(A 16) $$Q_2(\pm a) = \{(y_1, y_2, v_3, v_4):$$

$$y_1 = \delta_{12} \text{ satisfies (6.1) with } \rho_4 \text{ in place of } \rho_3,$$

$$\eta_1(\delta_{12}) \pm a < y_2 < \eta_2(\delta_{12}) \pm a,\ \eta_3(\delta_{12})$$

$$+ \eta_5 y_2 < v_3 < \eta_4(\delta_{12}) + \eta_5 y_2,$$

$$\eta_3^*(\delta_{12}) + \eta_5 y_2 < v_3 < \eta_4^*(\delta_{12}) + \eta_5 y_2;$$

$$|v_3 - v_4| < k(\rho_3^2 + \rho_4^2)^{1/2}\}$$

where $\eta_1 - \eta_5$ are as in (6.4) and η_3^*, η_4^* are the same as η_3, η_4 but with ρ_4 substituted for ρ_3. Now, consider this set and the remark following Lemma 4.2. Expression (A 16) can be rewritten as

$$Q_2(\pm a) = \{(y_1, y_2, v_3, v_4): \ y_1 = \delta_{12}$$

satisfies the appropriate version of (6.1),

$$\eta_1(\delta_{12}) \pm a < y_2 < \eta_2(\delta_{12}) \pm a$$

$$(v_3, v_4) \in R^*(\eta_5(y_2 - (\eta_1(\delta_{12}) + \eta_2(\delta_{12})/2); \delta_{12})\}$$

where

(A 17) $$R^*(t, \delta_{12}) = \{(v_3, v_4): \xi_1(\delta_{12}) < v_3 - t < \xi_2(\delta_{12}),\ \xi_3(\delta_{12}) < v_4 - t < \xi_4(\delta_{12}),\ |v_3 - v_4| < k(\rho_3^2 + \rho_4^2)^{1/2}\}.$$

(Here $\xi_1, \ldots, \xi_4$ could be expressed explicitly in terms of the preceding constants.) According to (4.7) it will follow that

(A 18) $$\Pr(Q_2(-a)) > \Pr(Q_2(+a))$$

if

(A 19) $$\Pr(R^*(-t, \delta_{12})) > \Pr(R^*(t, \delta_{12}))$$

for the appropriate values of δ_{12} and $t > 0$.

Now observe that $R^*(\pm t, \delta_{12})$ are sets of the same qualitative form as $S(\pm t)$, defined in (2.6) and treated in detail in Section 5. The only difference is that the constants $\xi_1 - \xi_4$ have different explicit expressions than the corresponding terms in the definition (2.6) of $S(\pm a)$. If it were not for this difference then (A 19) would have already been proven in Section 5 and the desired (A 18) would immediately follow.

Perhaps, however, (A 19) can be proven by the same steps as those of Section 5. Better yet, perhaps the setup in Section 5 can be weakened to prove a correspondingly stronger result which includes both Lemmas 5.1 and 5.2 and (A 19). If so then (A 18) would follow by this stronger result and induction. If this could be done for all the sets $Q_{j(i)}$ then Conjecture 1 would follow for $p = 6$.

One might even hope that the above remarks could form the basis of an inductive proof of Conjecture 1 for all p. The master key to constructing such a proof would of course be to formulate the appropriate stronger version referred to above of the results in Section 5.

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8. Miller, R. G. (1977). Developments in multiple comparisons, 1966-1976. Jour. Amer. Statist. Assoc. 72 779-788.

Corrections to "A proof that Kramer's Multiple Comparison Procedure is Level-α", revised December 1979.

1. The reference listed as Dunnett (1979) has now appeared. It should now be referenced as,

Dunnett, C. W. (1980). Pairwise multiple comparisons in the homogeneous variance, unequal sample size case. Jour. Amer. Statist. Assoc. 75, 789-795.

It should thus be referred to in the manuscript as Dunnett (1980) instead of Dunnett (1979).

2. The procedure should be referred to throughout as the Tukey-Kramer method. For example, the title should read "A proof that the Tukey-Kramer multiple comparison procedure ... etc."

3. p.13, line 3 should read: section; as well as $p = 4$ ... etc.

4. The following acknowledgment should be added on p.40:

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